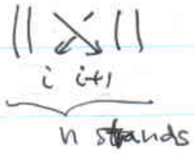
Cautis

Last time : we learn that the functor $\underbrace{\text{associated}}_{D(Y_n) \to D(Y_n)}$ to



is the Fourier-Mukai transform in the kernel

$$T_n^i(2) = \mathcal{O}_{Z_n^i} \otimes \pi_1^*(\mathcal{E}_{i+1}^\vee) \otimes \pi_2^*(\mathcal{E}_i)[-1]\{3\}$$

$$Z_n^i = \{(L_\cdot, L'_\cdot) \mid L_j = L'_j \text{ for } j \neq i\} \subset Y_n \times Y_n$$

$$Y_n \xleftarrow{\pi_1} Y_n \times Y_n \xrightarrow{\pi_2} Y_n$$

$L_{i+1} = L'_{i+1}$, L_i, L'_i, $L_{i-1} = L'_{i-1}$

$\Delta \cup V_n^i$ irreducible

diagonal

L_{i+1}, $L_i \xrightarrow{z} L_{i-1}$, L_i

this talk :

1. give another description of $T_n^i(2)$ in terms of spherical twists
2. (sketch) why this homology is iso. to Khovanov homology

Notation

$\mathcal{P} \in D^b(X \times Y)$ (complex of) sheaf

induces a FM transform $\Phi_{\mathcal{P}} : D(X) \to D(Y)$

$$\mathcal{F} \longmapsto \pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{P})$$

Facts 1) $\underset{\text{Right adjoint}}{(\Phi_{\mathcal{P}})_R} = \Phi_{(\mathcal{P}_R)}$ $\qquad \mathcal{P}_R = \mathcal{P}^\vee \otimes \pi_1^* \omega_X [\dim X] \in D(Y \times X)$

$(\Phi_{\mathcal{P}})_L = \Phi_{(\mathcal{P}_L)}$ $\qquad \mathcal{P}_L = \mathcal{P}^\vee \otimes \pi_2^* \omega_Y [\dim Y] \in$ 〃

2) $\mathcal{P}\in D(X\times Y)$, $Q\in D(Y\times Z)$

$$\Phi_Q\circ\Phi_{\mathcal{P}} = \Phi_{Q*\mathcal{P}} \qquad Q*\mathcal{P}=\pi_{13*}(\pi_{12}^*\mathcal{P}\otimes\pi_{23}^*Q)$$

$$\begin{tikzcd} & X\times Y\times Z \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X\times Y & X\times Z & Y\times Z \end{tikzcd}$$

Twist $\quad A\xrightarrow{\Phi} B \quad$ functors between triang. categories A, B

Φ induces <u>twist</u> $\quad T_\Phi \circlearrowright B$

$$\begin{array}{ccc} \mathcal{F} & \longmapsto & \mathrm{Cone}(\Phi\circ\Phi_R(\mathcal{F})\xrightarrow{\alpha}\mathcal{F}) \\ \cap & & \\ B & & \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}(\Phi\Phi_R(\mathcal{F}),\mathcal{F}) & \cong & \mathrm{Hom}(\Phi_R(\mathcal{F}),\Phi_R(\mathcal{F})) \\ \psi & & \psi \\ \alpha & \longleftarrow & \mathrm{id} \end{array}$$

in our case $\quad D(X)\xrightarrow{\Phi_{\mathcal{P}}} D(Y)$

$T_{\mathcal{P}}\curvearrowright D(Y)$ is the FM transform w.r.t.

$$\mathrm{Cone}(\mathcal{P}*\mathcal{P}_R\longrightarrow\mathcal{O}_{\Delta_Y}) =: T_{\mathcal{P}}\in D(Y\times Y)$$

<u>Q</u>. When is this twist an equivalence?

<u>A</u>. (partial) When $\Phi_{\mathcal{P}}$ is a spherical functor.

<u>Def</u>. $\quad D(X)\xrightarrow{\Phi_{\mathcal{P}}} D(Y) \quad$ Suppose

1) $\mathcal{P}_R\cong\mathcal{P}_L[K] \qquad K=\dim X-\dim Y$

2) in general $\exists$ a sequence of maps

$$D(X\times X) \ni \mathcal{O}_\Delta \longrightarrow \mathcal{P}_R * \mathcal{P}$$

$$\mathcal{P}_R * \mathcal{P} \cong \mathcal{P}_L * \mathcal{P}[K] \longrightarrow \mathcal{O}_\Delta[K] \qquad (\Delta)$$

$\mathcal{P}_R * \mathcal{P} \doteqdot \mathcal{O}_\Delta \oplus \mathcal{O}_\Delta[K]$

spherical

we suppose (Δ) is an exact triangle.

3) $\mathrm{Hom}^i(\mathcal{P}, \mathcal{P}) = \begin{cases} \mathbb{C} & \text{if } i=0 \\ 0 & \text{if } i=K, K+1 \end{cases}$

then $\mathcal{P}$ (or $\Phi_\mathcal{P}$) is a spherical functor

Thm. $\mathcal{P}$: spherical functor $\Rightarrow T_\mathcal{P}$ is an equivalence.

(due to Horja & Rouquier)

ex. $X = pt$ $\quad D(pt) \xrightarrow{\Phi_\mathcal{P}} D(Y)$ then $\Phi_\mathcal{P}$ is a spherical functor

$\mathbb{C} \longmapsto \mathcal{E}$ $\quad \Updownarrow$

$\mathcal{E}$: spherical object (Seidel–Thomas)

e.g. $D(Y_{n-2}) \xrightarrow{\Phi_{\mathcal{g}_n^i}} D(Y_n)$

$|| \cdots \cap \cdots ||$ (i, $i+1$)

$$\begin{array}{ccc} X_n^i & \overset{j}{\hookrightarrow} & Y_n \quad \text{Codim } 1 \\ \mathbb{P}^1\text{-bdle} \downarrow q & & \\ Y_{n-2} & & \end{array}$$

$$\mathcal{g}_n^i = \mathcal{O}_{X_n^i} \otimes \pi_2^* \mathcal{E}_i\{-i+1\}$$

$$\begin{array}{ccc} & Y_{n-2}\times Y_n & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ Y_{n-2} & & Y_n \end{array}$$

$\Phi_{\mathcal{g}_n^i}$ is spherical

<u>Thm</u> $T_n^i(2) = T_{\mathcal{G}_n^i}[-1]\{1\}$ So $T_n^i(2)$ is an equivalence

Why (sketch)?

$T_n^i(2) \leftrightarrow$ Kernel $\mathcal{O}_{Z_n^i}(\mathcal{L})[-1]$ $\quad Z_n^i = \Delta \cup V_n^i$

$T_{\mathcal{G}_n^i} \leftrightarrow$ Kernel Cone $(\mathcal{G}_n^i * (\mathcal{G}_n^i)_R \to \mathcal{O}_\Delta)$

$$\mathcal{O}_\Delta(-V_n^i) \to \mathcal{O}_{Z_n^i} \to \mathcal{O}_{V_n^i} \quad \triangle$$

$$\mathcal{O}_{Z_n^i} = \text{Cone}(\mathcal{O}_{V_n^i}[-1] \to \mathcal{O}_\Delta(-\mathcal{L}))$$

$$\therefore \mathcal{O}_{Z_n^i}(\mathcal{L}) = \text{Cone}(\mathcal{O}_{V_n^i}(\mathcal{L})[-1] \to \mathcal{O}_\Delta)$$

$$\mathcal{G}_n^i * (\mathcal{G}_n^i)_R = \mathcal{O}_{V_n^i}(\mathcal{L})[-1]$$ calculation

$i \quad i+1$

<u>Notice</u>  $\leftrightarrow$ Ker $F_n^i = (G_n^i)_R[1]\{-1\}$

calculation

 Kernel G_n^i

$\Rightarrow$  Ker $G_n^i * (G_n^i)_R[1]\{-1\}$

 Kernel $\mathcal{O}_\Delta$

 Kernel $\mathrm{Cone}\,(G_n^i * (G_n^i)_R \to \mathcal{O}_\Delta)[-1]\{1\}$

modulo shifts $\Psi($  $) \to \Psi(||) \to \Psi($  $)$ long exact sequence

NB. $\mathcal{O}_\Delta \to \beta$ $\mathcal{O}_\Delta[k]$ ...  $= \mathbb{C} \oplus \mathbb{C}[2]$

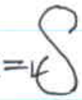
Thm $H^{i,j}(\Psi(K)(\mathbb{C})) = H^{i,j}_{alg}(K) = H^{i+j,j}_{Kh}(K)$

(sketch)

By definition $\Psi(K) = \mathcal{P}_1 * \mathcal{P}_2 * \cdots * \mathcal{P}_k$

$$\mathcal{P}_j \in D(Y_{n_{j-1}} \times Y_{n_j})$$

$\mathcal{P}_j = G_n^i$ or F_n^i or $\mathrm{Cone}(G_n^i * F_n^i \to \mathcal{O}_\Delta)$ modulo shifts

e.g. $K=$  $\Psi(K) = F_2^1 * T_2^1 * G_2^1$

basic properties

$$\Psi(K) = \pi_{0k}(\pi_{01}^* \mathcal{P}_1 \otimes \pi_{12}^* \mathcal{P}_2 \otimes \cdots \otimes \pi_{k-1,k}^* \mathcal{P}_k)$$

$$\begin{array}{c} Y_{n_0} \times Y_{n_1} \times \cdots \times Y_{n_k} \\ \swarrow \pi_{j-1,j} \\ Y_{n_{j-1}} \times Y_{n_j} \end{array}$$

Fact $\quad \mathrm{Cone}(G_j \to D_j) = E_j$, then

$E_1 \otimes \cdots \otimes E_j$ is isomorphic to the cone of a giant cone

Ex $\quad E_1 \otimes E_2 = \mathrm{Cone}\left(\begin{array}{ccc} G_1\otimes G_2 & \to & G_1 \otimes D_2 \\ \downarrow & & \downarrow \\ D_1 \otimes G_2 & \to & D_1 \otimes D_2 \end{array}\right)$

hence $\Psi(K) = \pi_{0K*}$ (cone of giant cube)

where on vertices

you see $\pi_{01}^* Q_1 \otimes \cdots \otimes \pi_{K-1,K}^* Q_K$

where $Q_k = G_n^i$ or F_n^i

but $\pi_{0K*}(\pi_{01}^* Q_1 \otimes \cdots \otimes \pi_{K-1,K}^* Q_K)$

$= \Psi(K_\delta)$

⋮ a resolution of K

$\mathrm{Cone}(A \xrightarrow{f} B \xrightarrow{g} C)$

$$\begin{array}{ccccc} A & \to & B & \to & \mathrm{Cone} \\ & & \downarrow & \rho & \downarrow \\ & & C & \xrightarrow{\sim} & C \\ & & & & \downarrow \\ & & & & \mathrm{cone} \end{array}$$

...... uniquely exists under homological condition

$F \subset U_n \subset Y_n$

$\cup$

X_α

⋮ corresponding to matching α

$\bigoplus_\alpha \mathcal{O}_{X_\alpha} = \mathcal{F}$

Conj. $\quad H_n = \mathrm{Ext}^*(\mathcal{F}, \mathcal{F})$

$D_F^b(\mathcal{U}) \cong D(H_n\text{-mod.})$